Adjoint Method in PDE-based Image Compression

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Organization

Goal: Reconstruct a missing part of an image $f : K \subseteq D \to [0,1]$ given.

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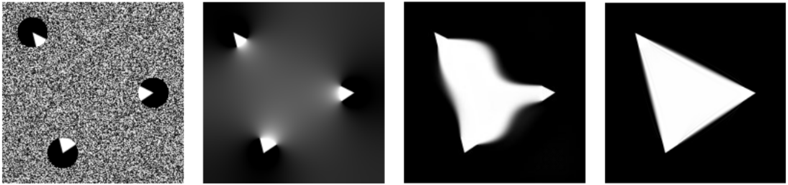
Inpainting

Find $u : D \to [0,1]$ such that :

$$\begin{cases} A(u) = 0 & \text{in } D \setminus K, \\ u = f & \text{in } K, \\ \frac{\partial u}{\partial \mathrm{n}} = 0 & \text{on } \partial D, \end{cases}$$

with D being the support of the image, and A an operator (PDE).



Heat equation

Figure: Some examples of image inpainting.

Heat equation:

$$\begin{cases} \partial_t u = \Delta u & \text{in } [0,+\infty[\times D \setminus K, \\ u = f & \text{in } [0,+\infty[\times K, \\ \frac{\partial u}{\partial \mathrm{n}} = 0 & \text{on } [0,+\infty[\times \partial D, \end{cases}$$

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Time discretization: $\partial_t u \approx \frac{u_{n+1} - u_n}{\alpha}$,

$$\begin{cases} u - \alpha \Delta u = u_0 & \text{in } D \setminus K, \\ u = f & \text{in } K, \\ \frac{\partial u}{\partial \mathrm{n}} = 0 & \text{on } \partial D. \end{cases}$$

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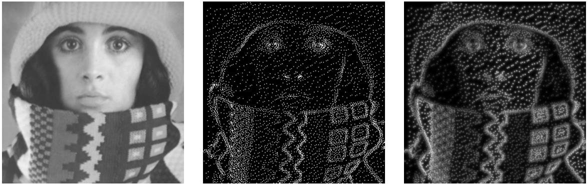

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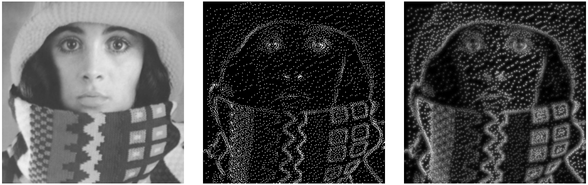
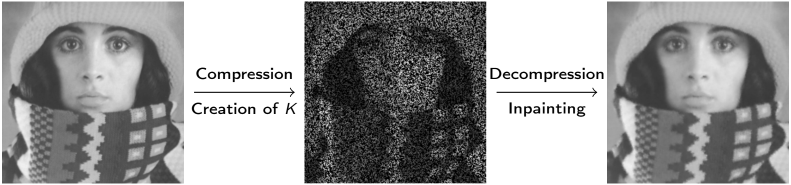
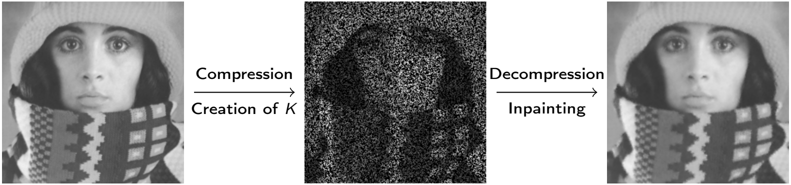




Original image f. Compressed image. Decompressed image u.

Figure: Compression by inpainting.



Original image f. Compressed image. Decompressed image u.

Figure: Compression by inpainting.

Question

Which pixels to keep?

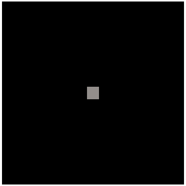
Shape optimization problem:

$$\min_{K\subseteq D,\ m(K)\leq c} \left\{ \mathcal{E}(u_K) \;\middle|\; u_K \text{ solution of the inpainting} \right\},$$

with

- $\mathcal{E}$ the error between the decompressed image and the original image,
- m quantifies the size of K.

Example of the importance of the choice of K: Only one pixel.



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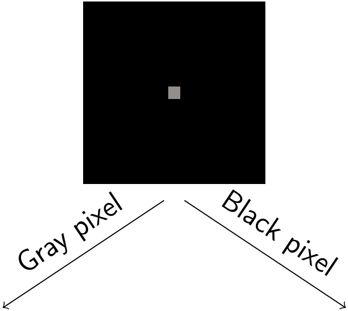

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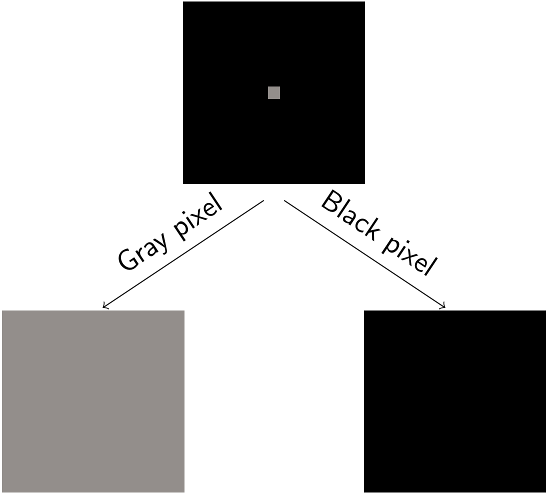

$\mathcal{E}(u_K)$ large.

$\mathcal{E}(u_K)$ small + denoising.

If we consider that the image is noisy, we can remove this noise by compressing.

Original

Impulse noise

Gaussian noise

Figure: Example of noise.

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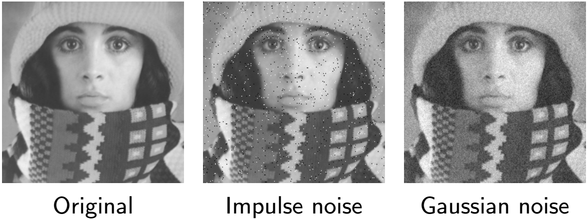

Figure: Example of noise.

Classic in image processing: for $\alpha > 0$,

$$\min_u \{\mathcal{D}(u, f) + \alpha \mathcal{R}(u)\}.$$

Compression problem

For $1 \leq p \leq 2$,

$$\min_{K \subseteq D,\ m(K) \leq c} \left\{ \frac{1}{p} \int_D |u_K - f|^p \, dx \;\middle|\; u_K \text{ solution of (1)} \right\},$$

where

$$\begin{cases} u - \alpha \Delta u = u_0 & \text{in } D \setminus K, \\ u = f & \text{in } K, \\ \frac{\partial u}{\partial \mathrm{n}} = 0 & \text{on } \partial D. \end{cases} \tag{1}$$

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Note:

- compression step: $u_0 = f$, decompression step: $u_0 = 0$.

Organization

1. Modeling
2. Analysis
3. Numerical

Question

How to create K?

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How to create K?

Definition (Topological gradient)

Let $x_0 \in D$ and $K_\varepsilon = K \cup \overline{B(x_0, \varepsilon)}$.

$$\mathcal{E}(u_{K_\varepsilon}) - \mathcal{E}(u_K) = \rho(\varepsilon)\, G(x_0) + o\big(\rho(\varepsilon)\big),$$

where

- ρ is a positive function going to zero as ε approaches zero,
- G is the so-called topological gradient.

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- ρ is a positive function going to zero as ε approaches zero,
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To minimize the cost functional, one has to create small holes at the locations x_0 where $G(x_0)$ is the most negative.

Setting $\widetilde{v}_K = u_K - f$, we can write equivalently,

Problem

Find $\widetilde{v}_K$ in $H^1(D)$ such that

$$\begin{cases} -\alpha\Delta\widetilde{v}_K + \widetilde{v}_K = \alpha\Delta f & \text{in } D \setminus K, \\ \widetilde{v}_K = 0 & \text{in } K, \\ \frac{\partial \widetilde{v}_K}{\partial \mathrm{n}} = 0 & \text{on } \partial D. \end{cases}$$

For the analysis: $K_\varepsilon := B(x_0, \varepsilon)$, $\widetilde{v}_\varepsilon := \widetilde{v}_{K_\varepsilon}$ and $\widetilde{v}_0 := \widetilde{v}_\emptyset$.

Weak formulation: find $\widetilde{v}_\varepsilon$ in V_ε such that,

$$a_\varepsilon(\widetilde{v}_\varepsilon, \varphi) = l_\varepsilon(\varphi), \ \forall \varphi \in V_\varepsilon,$$

with

$$V_\varepsilon := \{v \in H^1(D \setminus B_\varepsilon) \mid v = 0 \text{ on } \partial B_\varepsilon\},$$

$$a_\varepsilon(\widetilde{v}_\varepsilon, \varphi) := \alpha \int_{D \setminus B_\varepsilon} \nabla \widetilde{v}_\varepsilon \cdot \nabla \varphi \, dx + \int_{D \setminus B_\varepsilon} \widetilde{v}_\varepsilon \, \varphi \, dx,$$

$$l_\varepsilon(\varphi) := \int_{D \setminus B_\varepsilon} h \, \varphi \, dx.$$

we suppose that there exists

- a continuous bilinear form $\delta a : V \times V \to \mathbb{R}$,
- a continuous linear form $\delta l : V \to \mathbb{R}$,
- a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $\varepsilon \geq 0$,

such that

- $\|a_\varepsilon - a_0 - \rho(\varepsilon)\,\delta a\|_{\mathcal{L}_2(V)} = o\big(\rho(\varepsilon)\big)$,
- $\|l_\varepsilon - l_0 - \rho(\varepsilon)\,\delta l\|_{\mathcal{L}(V)} = o\big(\rho(\varepsilon)\big)$,
- $\lim_{\varepsilon \to 0} \rho(\varepsilon) = 0$.

Theorem

Let $\widetilde{v}_\varepsilon \in V$ be the solution of the following problem : find $\widetilde{v} \in V$ such that,

$$a_\varepsilon(\widetilde{v}, \varphi) = l_\varepsilon(\varphi), \ \forall \varphi \in V.$$

Let $\widetilde{w_0}$ be the solution of the so-called adjoint problem: find $\widetilde{w} \in V$ such that

$$a_0(\widetilde{w}, \varphi) = -D\mathcal{E}(\widetilde{v_0})\varphi = -\int_D \widetilde{v_0}|\widetilde{v_0}|^{p-2}\varphi \, dx, \ \forall \varphi \in V.$$

Then,

$$\mathcal{E}(\widetilde{v_\varepsilon}) - \mathcal{E}(\widetilde{v_0}) = \rho(\varepsilon)\big(\delta a(\widetilde{v_0}, \widetilde{w_0}) - \delta l(\widetilde{w_0})\big) + o\big(\rho(\varepsilon)\big).$$

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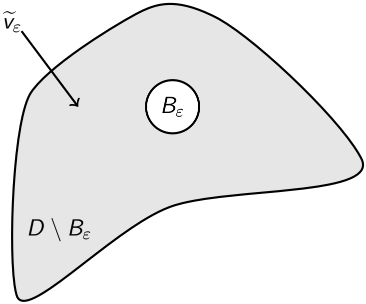
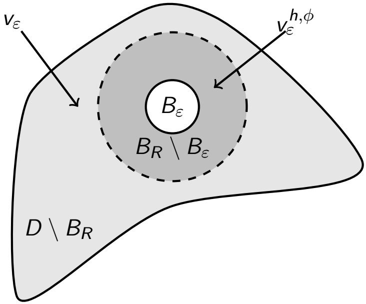
Problem: our $V = V_\varepsilon := \{v \in H^1(D \setminus B_\varepsilon) \mid v = 0 \text{ on } \partial B_\varepsilon\}$ depends on ε.

Truncation technique :



Before splitting.



After splitting.

Proposition

We have

$$\widetilde{v}_\varepsilon = \begin{cases} v_\varepsilon & \text{in } D \setminus B_R, \\ v_\varepsilon^{h,\phi} & \text{in } B_R \setminus B_\varepsilon. \end{cases}$$

For v, φ in $V_R := H^1(D \setminus B_R)$, we define

$$a_\varepsilon(v,\varphi) := \alpha \int_{D\setminus B_R} \nabla v \cdot \nabla \varphi \; dx + \alpha \int_{\partial B_R} \partial_n v_\varepsilon^{0,\phi} \, \varphi \; d\sigma + \int_{D\setminus B_R} v \, \varphi \; dx,$$

$$l_\varepsilon(\varphi) := \int_{D\setminus B_R} h \, \varphi \; dx - \alpha \int_{\partial B_R} \partial_n v_\varepsilon^{h,0} \, \varphi \; d\sigma.$$

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Now V depends no longer on ε!
We need to compute δa and δl.

$$a_\varepsilon(v,\varphi) := \alpha \int_{D\backslash B_R} \nabla v \cdot \nabla \varphi \; dx + \alpha \int_{\partial B_R} \partial_n v_\varepsilon^{0,\phi} \, \varphi \; d\sigma + \int_{D\backslash B_R} v \, \varphi \; dx,$$

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$$l_\varepsilon(\varphi) := \int_{D\setminus B_R} h\, \varphi \, dx - \alpha \int_{\partial B_R} \partial_n v_\varepsilon^{h,0} \, \varphi \, d\sigma.$$

δa and δl

$$a_\varepsilon(v,\varphi) - a_0(v,\varphi) = \alpha \int_{\partial B_R} \partial_n (v_\varepsilon^{0,\phi} - v_0^{0,\phi}) \, \varphi \, d\sigma,$$

$$l_\varepsilon(\varphi) - l_0(\varphi) = -\alpha \int_{\partial B_R} \partial_n (v_\varepsilon^{h,0} - v_0^{h,0}) \, \varphi \, d\sigma.$$

For $v_\varepsilon^{0,\phi}$:

$$\begin{cases} -\alpha\Delta v_\varepsilon^{0,\phi} + v_\varepsilon^{0,\phi} = 0 & \text{in } B_R \setminus B_\varepsilon, \\ v_\varepsilon^{0,\phi} = 0 & \text{on } \partial B_\varepsilon, \\ v_\varepsilon^{0,\phi} = \phi & \text{on } \partial B_R. \end{cases}$$

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Proof.

- Using polar coordinates in $\mathbb{R}^2$, we have,

$$v_\varepsilon^{0,\phi}(r,\theta) = \sum_{n\in\mathbb{Z}} c_n(r)\, e^{in\theta},$$

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$$v_\varepsilon^{0,\phi}(r,\theta) = \sum_{n\in\mathbb{Z}} c_n(r)\, e^{in\theta},$$

- where c_n satisfies, for all n in $\mathbb{Z}$, and $0 < r \leq R$,

$$-\alpha\, r^2\, c_n''(r) - \alpha\, r\, c_n'(r) + (r^2 + \alpha n^2) c_n(r) = 0.$$

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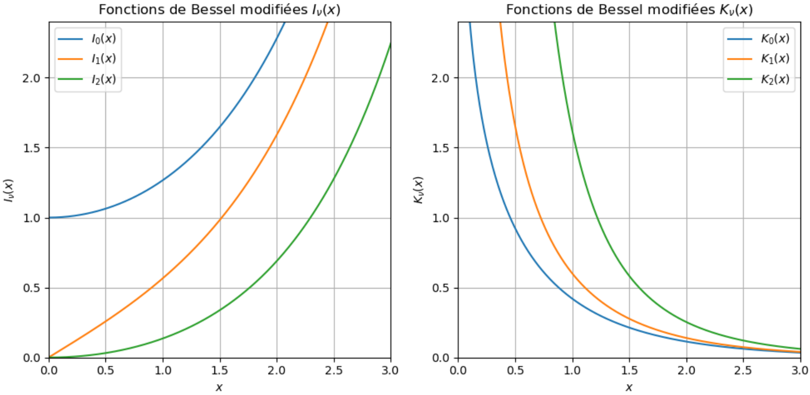
- We solve the equation using Bessel functions. □


Fonctions de Bessel modifiées $I_\nu(x)$
$I_0(x)$
$I_1(x)$
$I_2(x)$
$I_\nu(x)$
x
Fonctions de Bessel modifiées $K_\nu(x)$
$K_0(x)$
$K_1(x)$
$K_2(x)$
$K_\nu(x)$
x

Proposition

For ϕ in $H^{1/2}(\partial B_R)$, we set,

$$\delta v^{0,\phi}(r) := -\phi_0 \frac{I_0(\alpha^{-1/2}R)K_0(\alpha^{-1/2}r) - K_0(\alpha^{-1/2}R)I_0(\alpha^{-1/2}r)}{I_0(\alpha^{-1/2}R)^2}.$$

Then, for ε sufficiently small, we have the following asymptotic estimation,

$$v_\varepsilon^{0,\phi}(r,\theta) - v_0^{0,\phi}(r,\theta) - \frac{-1}{\ln \varepsilon}\delta v^{0,\phi}(r) = o\left(\frac{-1}{\ln \varepsilon}\right).$$

For $a_\varepsilon(v, \varphi) - a_0(v, \varphi)$:

Proof.

$$a_\varepsilon(v, \varphi) - a_0(v, \varphi) = \alpha \int_{\partial B_R} \partial_n(v_\varepsilon^{0,\phi} - v_0^{0,\phi})\, \varphi \; d\sigma$$

□

Proposition

For ϕ in $H^{1/2}(\partial B_R)$, we define,

$$\delta a(v, w) := \alpha v(x_0) w(x_0).$$

Then, for ε sufficiently small, we have the following asymptotic estimation, $\forall v, w \in V_R$:

$$\left| a_\varepsilon(v, w) - a_0(v, w) - \frac{-2\pi}{\ln \varepsilon} \delta a(v, w) \right| = o\left(\frac{-1}{\ln \varepsilon}\right) \|v\|_{V_R} \|w\|_{V_R}.$$

For $v_\varepsilon^{h,0}$:

$$\begin{cases} -\alpha\Delta v_\varepsilon^{h,0} + v_\varepsilon^{h,0} = h & \text{in } B_R \setminus B_\varepsilon, \\ v_\varepsilon^{h,0} = 0 & \text{on } \partial B_\varepsilon, \\ v_\varepsilon^{h,0} = 0 & \text{on } \partial B_R. \end{cases}$$

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Proof.

- Using polar coordinates in $\mathbb{R}^2$, we have,

$$v_\varepsilon^{h,0}(r,\theta) = \sum_{n\in\mathbb{Z}} c_n(r)\, e^{in\theta} \quad \text{and} \quad h(r,\theta) = \sum_{n\in\mathbb{Z}} h_n(r)\, e^{in\theta},$$

- where c_n satisfies, for all n in $\mathbb{Z}$ and $0 < r \leq R$,

$$-\alpha\, r^2\, c_n''(r) - \alpha\, r\, c_n'(r) + (r^2 + \alpha n^2) c_n(r) = r^2 h_n(r).$$

- We solve the equation using Bessel functions. □

Proposition

For ε small enough, we have,

$$\mathcal{E}(K_\varepsilon) - \mathcal{E}(K) = \underbrace{4\pi\alpha\, \widetilde{v}_0(x_0)\, \widetilde{w}_0(x_0)}_{=G(x_0)} \frac{-1}{\ln \varepsilon} + o\left(\frac{-1}{\ln \varepsilon}\right),$$

with

$$\begin{cases} -\alpha\Delta\widetilde{v}_0 + \widetilde{v}_0 = \alpha\Delta f & \text{in } D, \\ \partial_n \widetilde{v}_0 = 0 & \text{on } \partial D, \end{cases}$$

and

$$\begin{cases} -\alpha\Delta\widetilde{w}_0 + \widetilde{w}_0 = -\widetilde{v}_0|\widetilde{v}_0|^{p-2} & \text{in } D, \\ \partial_n \widetilde{w}_0 = 0 & \text{on } \partial D. \end{cases}$$

Question

Which pixels to keep?

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Which pixels to keep?

Answer

We have to keep the pixels x_0 which minimize the product $v_0(x_0)w_0(x_0)$.

Organization

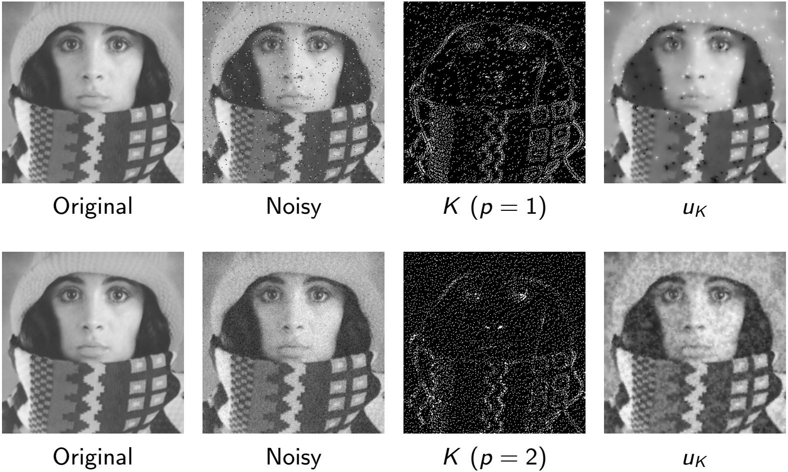

Figure: 10% of the total number of pixels.

Conclusion:

- Another operator,
- Difficulty in saving K in practice,
- Not great for Gaussian noise (there are empirical techniques).

Thank you for your attention!

[1] Zakaria Belhachmi and Thomas Jacumin. "Adjoint method in PDE-based image compression". In: Asymptotic Analysis (Oct. 2024), pp. 1–28. ISSN: 18758576, 09217134. DOI: 10.3233/ASY-241944.