# Bachelor Thesis – Adaptive Finite Differences Method (AFDM)

Thomas Jacumin

March 2024

The aim is to study an adaptive finite differences scheme as presented in the paper of Oberman and Zwiers [5]. The difference here is that our *dofs* (degrees of freedom) are located in the middle of the elements and not on the corners, allowing us to mimic the piecewise constant functions space found in the finite element methods. We consider a quad-mesh and add constraints on it such that our adaptive finite differences can be computed at every node of the mesh [2, Chapter 14]. We denote in the following $dx, dy > 0$ the width and the height of an element.

We use the usual finite differences for regular nodes [4, 3]. We set $x_{\mathrm{E}} = x + dx$ and $y_{\mathrm{E}} = y$ (see Figure 1).

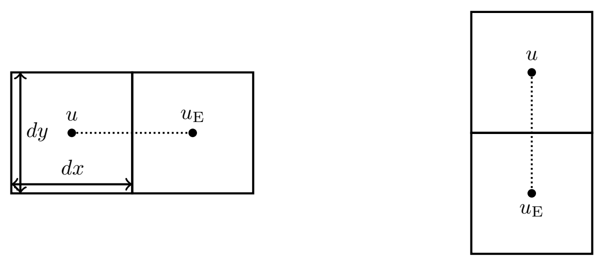


Figure 1: Illustration of a regular node with $u := u(x, y)$, $u_{\mathrm{E}} := u(x_{\mathrm{E}}, y_{\mathrm{E}})$.

We apply a Taylor expansion [6, Appendix A] around $(x, y)$, and we get,

$$u(x_{\mathrm{E}}, y_{\mathrm{E}}) = u(x, y) + dx\, \partial_x u(x, y) + O(dx^2).$$

Then, we have the forward $x$-derivative of $u$ at $(x, y)$, given by,

$$\partial_x u(x, y) = \frac{u(x_{\mathrm{E}}, y_{\mathrm{E}}) - u(x, y)}{dx} + O(dx).$$

Similarly, we have the forward $y$-derivative of $u$ at $(x, y)$,

$$\partial_y u(x, y) = \frac{u(x_{\mathrm{S}}, y_{\mathrm{S}}) - u(x, y)}{dy} + O(dy).$$

Now, we deal with the 3 different cases for the dangling nodes. Start with the first one, that we call dangling 1. For convenience, we set $x_{\mathrm{NE}} = x_{\mathrm{SE}} := x + \frac{3}{4}dx$, $y_{\mathrm{NE}} := y - \frac{1}{4}dy$, and $y_{\mathrm{SE}} := y + \frac{1}{4}dy$ (see Figure 2).

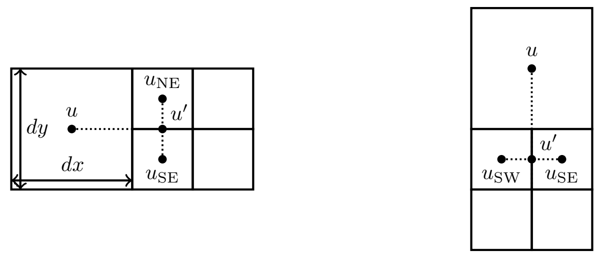


Figure 2: Illustration of a dangling node 1 with $u := u(x,y)$, $u_{\mathrm{NE}} := u(x_{\mathrm{NE}}, y_{\mathrm{NE}})$, $u_{\mathrm{SE}} := u(x_{\mathrm{SE}}, y_{\mathrm{SE}})$ and $u' = \frac{1}{2}(u_{\mathrm{NE}} + u_{\mathrm{SE}})$.

We apply a Taylor expansion, and we get,

$$u(x_{\mathrm{NE}}, y_{\mathrm{NE}}) = u(x,y) + \frac{3}{4} dx\, \partial_x u(x,y) - \frac{1}{4} dy\, \partial_y u(x,y) + O(dx^2 + dy^2),$$

$$u(x_{\mathrm{SE}}, y_{\mathrm{SE}}) = u(x,y) + \frac{3}{4} dx\, \partial_x u(x,y) + \frac{1}{4} dy\, \partial_y u(x,y) + O(dx^2 + dy^2).$$

We sum,

$$u(x_{\mathrm{NE}}, y_{\mathrm{NE}}) + u(x_{\mathrm{SE}}, y_{\mathrm{SE}}) = 2u(x,y) + 2\frac{3}{4} dx\, \partial_x u(x,y) + O(dx^2 + dy^2)$$

Then, we have the forward $x$-derivative of $u$ at $(x,y)$, given by,

$$\partial_x u(x,y) = \frac{\frac{u(x_{\mathrm{NE}}, y_{\mathrm{NE}}) + u(x_{\mathrm{SE}}, y_{\mathrm{SE}})}{2} - u(x,y)}{(3/4)\, dx} + O\Big(dx + \frac{dy^2}{dx}\Big).$$

Similarly, we have the forward $y$-derivative of $u$ at $(x,y)$,

$$\partial_y u(x,y) = \frac{\frac{u(x_{\mathrm{SW}}, y_{\mathrm{SW}}) + u(x_{\mathrm{SE}}, y_{\mathrm{SE}})}{2} - u(x,y)}{(3/4)\, dy} + O\Big(\frac{dx^2}{dy} + dy\Big).$$

For the dangling node 2, we set $x_{\mathrm{N}} := x$, $y_{\mathrm{N}} := y - dy$, $x_{\mathrm{E}} := x + \frac{3}{2}dx$ and $y_{\mathrm{E}} := y - \frac{1}{2}dy$ (see Figure 3).

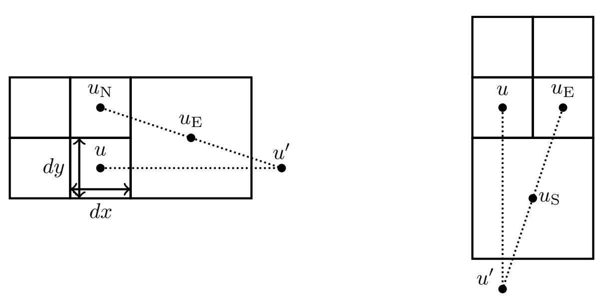


Figure 3: Illustration of a dangling node 2 with $u := u(x,y)$, $u_{\mathrm{E}} := u(x_{\mathrm{E}}, y_{\mathrm{E}})$, $u_{\mathrm{N}} := u(x_{\mathrm{N}}, y_{\mathrm{N}})$ and $u' = 2u_{\mathrm{E}} - u_{\mathrm{N}}$.

We apply a Taylor expansion, and we get,

$$u(x_{\mathrm{N}}, y_{\mathrm{N}}) = u(x,y) - dy\, \partial_y u(x,y) + O(dy^2),$$

$$u(x_{\mathrm{E}}, y_{\mathrm{E}}) = u(x,y) + \frac{3}{2} dx\, \partial_x u(x,y) - \frac{1}{2} dy\, \partial_y u(x,y) + O(dx^2 + dy^2).$$

We sum,

$$u(x_{\mathrm{N}}, y_{\mathrm{N}}) - 2u(x_{\mathrm{E}}, y_{\mathrm{E}}) = -u(x,y) - 3dx\, \partial_x u(x,y) + O(dx^2 + dy^2).$$

Then, we have the forward $x$-derivative of $u$ at $(x, y)$, given by,

$$\partial_x u(x, y) = \frac{2u(x_{\mathrm{E}}, y_{\mathrm{E}}) - u(x_{\mathrm{N}}, y_{\mathrm{N}}) - u(x, y)}{3\,dx} + O\Big(dx + \frac{dy^2}{dx}\Big).$$

Similarly, we have the forward $y$-derivative of $u$ at $(x, y)$,

$$\partial_y u(x, y) = \frac{2u(x_{\mathrm{S}}, y_{\mathrm{S}}) - u(x_{\mathrm{E}}, y_{\mathrm{E}}) - u(x, y)}{3\,dy} + O\Big(\frac{dx^2}{dy} + dy\Big).$$

For the dangling node 3, we set $x_{\mathrm{S}} := x$, $y_{\mathrm{S}} := y + dy$, $x_{\mathrm{E}} := x + \frac{3}{2}dx$ and $y_{\mathrm{E}} := y + \frac{1}{2}dy$ (see Figure 4).

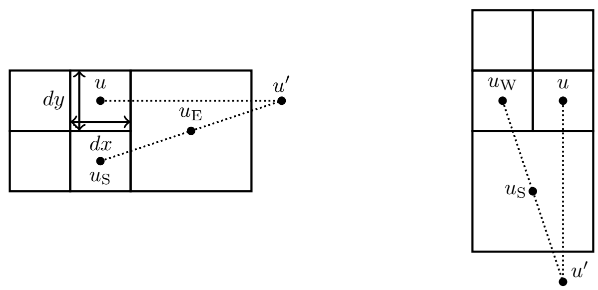


Figure 4: Illustration of a dangling node 3 with $u := u(x, y)$, $u_{\mathrm{E}} := u(x_{\mathrm{E}}, y_{\mathrm{E}})$, $u_{\mathrm{S}} := u(x_{\mathrm{S}}, y_{\mathrm{S}})$ and $u' = 2u_{\mathrm{E}} - u_{\mathrm{S}}$.

We apply a Taylor expansion, and we get,

$$u(x_{\mathrm{S}}, y_{\mathrm{S}}) = u(x, y) + dy\,\partial_y u(x, y) + O(dy^2),$$

$$u(x_{\mathrm{E}}, y_{\mathrm{E}}) = u(x, y) + \frac{3}{2}dx\,\partial_x u(x, y) + \frac{1}{2}dy\,\partial_y u(x, y) + O(dx^2 + dy^2).$$

We sum,

$$u(x_{\mathrm{S}}, y_{\mathrm{S}}) - 2u(x_{\mathrm{E}}, y_{\mathrm{E}}) = -u(x, y) - 3dx\,\partial_x u(x, y) + O(dx^2 + dy^2).$$

Then, we have the forward $x$-derivative of $u$ at $(x, y)$, given by,

$$\partial_x u(x, y) = \frac{2u(x_{\mathrm{E}}, y_{\mathrm{E}}) - u(x_{\mathrm{S}}, y_{\mathrm{S}}) - u(x, y)}{3\,dx} + O\Big(dx + \frac{dy^2}{dx}\Big).$$

Similarly, we have the forward $y$-derivative of $u$ at $(x, y)$,

$$\partial_y u(x, y) = \frac{2u(x_{\mathrm{S}}, y_{\mathrm{S}}) - u(x_{\mathrm{W}}, y_{\mathrm{W}}) - u(x, y)}{3\,dy} + O\Big(\frac{dx^2}{dy} + dy\Big).$$

# To Do

- Using Taylor expansion [6, Appendix A], find the 1d finite differences scheme of $u'(x)$ and $u''(x)$ (explain the Landau notation).
- Write [3] the discretization on uniform grid in 1d of

  $$-\alpha u''(x) + u(x) = f(x),$$

  with $\alpha > 0$.
- Implement it with DUNE [1] and solve it numerically.

- Write the discretization on uniform grid in 2d of

$$-\alpha \operatorname{div}(\nabla u) + u = f.$$

- Compute the $y$-derivative for the adaptive finite differences method.
- Create on DUNE an adaptive quad-mesh with small elements near the edges of $f$ (i.e. where $|\Delta f|$ is large).
- Implement the AFDM in DUNE and solve numerically

$$-\alpha \operatorname{div}(\nabla u) + u = f,$$

on the adaptive mesh.

# References


[1] M. Blatt, A. Burchardt, A. Dedner, Ch. Engwer, J. Fahlke, B. Flemisch, Ch. Gersbacher, C. Gräser, F. Gruber, Ch. Grüninger, D. Kempf, R. Klöfkorn, T. Malkmus, S. Müthing, M. Nolte, M. Piatkowski, and O. Sander. The Distributed and Unified Numerics Environment, Version 2.4. *Archive of Numerical Software*, 4(100):13–29, 2016.

[2] Mark de Berg, Marc van Krefeld, Mark Overmars, and Otfried Cheong. *Computational Geometry: Algorithms and Applications*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2nd ed. 2000 edition, 2000. OCLC: 1227482129.

[3] Hans Petter Langtangen and Svein Linge. *Finite Difference Computing with PDEs: A Modern Software Approach*, volume 16 of *Texts in Computational Science and Engineering*. Springer International Publishing, Cham, 2017.

[4] Zhilin Li, Zhonghua Qiao, and Tao Tang. *Numerical Solution of Differential Equations: Introduction to Finite Difference and Finite Element Methods*. Cambridge University Press, 1 edition, November 2017.

[5] Adam M. Oberman and Ian Zwiers. Adaptive Finite Difference Methods for Nonlinear Elliptic and Parabolic Partial Differential Equations with Free Boundaries. *Journal of Scientific Computing*, 68(1):231–251, July 2016.

[6] Endre Süli and D. F. Mayers. *An introduction to numerical analysis*. Cambridge University Press, Cambridge ; New York, 2003. OCLC: ocm50525488.